UNIFORM ESTIMATES FOR METASTABLE TRANSITION TIMES IN A COUPLED BISTABLE SYSTEM

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ABSTRACT. We consider a coupled bistable N-particle system on $\mathbb{R}^N$ driven by a Brownian noise, with a strong coupling corresponding to the synchronised regime. Our aim is to obtain sharp estimates on the metastable transition times between the two stable states, both for fixed N and in the limit when N tends to infinity, with error estimates uniform in N. These estimates are a main step towards a rigorous understanding of the metastable behavior of infinite dimensional systems, such as the stochastically perturbed Ginzburg-Landau equation. Our results are based on the potential theoretic approach to metastability.

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1. INTRODUCTION

The aim of this paper is to analyze the behavior of metastable transition times for a gradient diffusion model, independently of the dimension. Our method is based on potential theory and requires the existence of a reversible invariant probability measure. This measure exists for Brownian driven diffusions with gradient drift.

To be specific, we consider here a model of a chain of coupled particles in a double well potential driven by Brownian noise (see e.g. [2]). I.e., we consider the system of stochastic differential equations

$$\mathrm{d}X_\epsilon(t) = -\nabla F_{\gamma,N}(X_\epsilon(t))\mathrm{d}t + \sqrt{2\epsilon}\mathrm{d}B(t), \tag{1.1}$$

where $X_\varepsilon(t) \in \mathbb{R}^N$ and

$$F_{\gamma,N}(x) = \sum_{i\in\Lambda}\left(\frac{1}{4}x_i^4 - \frac{1}{2}x_i^2\right) + \frac{\gamma}{4}\sum_{i\in\Lambda}(x_i - x_{i+1})^2, \tag{1.2}$$

with $\Lambda = \mathbb{Z}/N\mathbb{Z}$ and $\gamma > 0$ is a parameter. B is a N dimensional Brownian motion and $\varepsilon > 0$ is the intensity of the noise. Each component (particle) of this system is subject to force derived from a bistable potential. The components of the system are coupled to their nearest neighbor with intensity γ and perturbed by independent noises of constant variance ε. While the system without noise, i.e. $\varepsilon = 0$, has several stable fixpoints, for $\varepsilon > 0$ transitions between these fixpoints will occur at suitable timescales. Such a situation is called metastability.

For fixed N and small ε, this problem has been widely studied in the literature and we refer to the books by Freidlin and Wentzell [9] and Olivieri and Vares [15] for further discussions. In recent years, the potential theoretic approach, initiated by Bovier, Eckhoff, Gayrard, and Klein [5] (see [4] for a review), has allowed to give very precise results on such transition times and notably led to a proof of the so-called Eyring-Kramers formula which provides sharp asymptotics for these transition times, for any fixed dimension. However, the results obtained in [5] do not include control of the error terms that are uniform in the dimension of the system.

Our aim in this paper is to obtain such uniform estimates. These estimates constitute a the main step towards a rigorous understanding of the metastable behavior of infinite dimensional systems, i.e. stochastic partial differential equations (SPDE) such as the stochastically perturbed Ginzburg-Landau equation. Indeed, the deterministic part of the system (1.1) can be seen as the discretization of the drift part of this SPDE, as has been noticed e.g. in [3]. For a heuristic discussion of the metastable behavior of this SPDE, see e.g. [13] and [17]. Rigorous results on the level of the large deviation asymptotics were obtained e.g. by Faris and Jona-Lasinio [10], Martinelli et al. [14], and Brassesco [7].

In the present paper we consider only the simplest situation, the so-called synchronization regime, where the coupling γ between the particles is so strong that there are only three relevant critical points of the potential $F_{\gamma,N}$ (1.2). A generalization to more complex situations is however possible and will be treated elsewhere.

The remainder of this paper is organized as follows. In Section 2 we recall briefly the main results from the potential theoretic approach, we recall the key properties of the potential $F_{\gamma,N}$, and we state the results on metastability that follow from the results of [5] for fixed N. In Section 3 we deal with the case when N tends to infinity and state our main result, Theorem 3.1. In Section 4 we prove the main theorem through sharp estimates on the relevant capacities.

In the remainder of the paper we adopt the following notations:

- for $t \in \mathbb{R}$, $\lfloor t \rfloor$ denotes the unique integer k such that $k \le t < k+1$;
- $\tau_D \equiv \inf\{t > 0 : X_t \in D\}$ is the hitting time of the set D for the process (X_t);
- $B_r(x)$ is the ball of radius $r > 0$ and center $x \in \mathbb{R}^N$;
- for $p \ge 1$, and $(x_k)_{k=1}^N$ a sequence, we denote the L^P-norm of x by

$$\|x\|_p = \left(\sum_{k=1}^{N} |x_k|^p \right)^{1/p}. \tag{1.3}$$

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2. Preliminaries

2.1. **Key formulas from the potential theory approach.** We recall briefly the basic formulas from potential theory that we will need here. The diffusion X_ϵ is the one introduced in (1.1) and its infinitesimal generator is denoted by L. Note that L is the closure of the operator

$$L = \varepsilon e^{F_{\gamma,N}/\varepsilon} \nabla e^{-F_{\gamma,N}/\varepsilon} \nabla. \tag{2.1}$$

For A, D regular open subsets of $\mathbb{R}^N$, let $h_{A,D}(x)$ be the harmonic function (with respect to the generator L) with boundary conditions 1 in A and 0 in D. Then, for $x \in (A \cup D)^c$, one has $h_{A,D}(x) = \mathbb{P}_x[\tau_A < \tau_D]$. The equilibrium measure, $e_{A,D}$, is then defined (see e.g. [8]) as the unique measure on ∂A such that

$$h_{A,D}(x) = \int_{\partial A} e^{-F_{\gamma,N}(y)/\varepsilon} G_{D^c}(x, y) e_{A,D}(dy), \tag{2.2}$$

where G_{D^c} is the Green function associated with the generator L on the domain D^c. This yields readily the following formula for the hitting time of D (see. e.g. [5]):

$$\int_{\partial A} \mathbb{E}_z[\tau_D] e^{-F_{\gamma,N}(z)/\varepsilon} e_{A,D}(dz) = \int_{D^c} h_{A,D}(y) e^{-F_{\gamma,N}(y)/\varepsilon} dy. \tag{2.3}$$

The capacity, $\mathrm{cap}(A, D)$, is defined as

$$\mathrm{cap}(A, D) = \int_{\partial A} e^{-F_{\gamma,N}(z)/\varepsilon} e_{A,D}(dz). \tag{2.4}$$

Therefore,

$$\nu_{A,D}(dz) = \frac{e^{-F_{\gamma,N}(z)/\varepsilon} e_{A,D}(dz)}{\mathrm{cap}(A, D)} \tag{2.5}$$

is a probability measure on ∂A, that we may call the equilibrium probability. The equation (2.3) then reads

$$\int_{\partial A} \mathbb{E}_z[\tau_D] \nu_{A,D}(dz) = \mathbb{E}_{\nu_{A,D}}[\tau_D] = \frac{\int_{D^c} h_{A,D}(y) e^{-F_{\gamma,N}(y)/\varepsilon} dy}{\mathrm{cap}(A, D)}. \tag{2.6}$$

The strength of this formula comes from the fact that the capacity has an alternative representation through the Dirichlet variational principle (see e.g. [11]),

$$\mathrm{cap}(A, D) = \inf_{h \in \mathcal{H}} \Phi(h), \tag{2.7}$$

where

$$\mathcal{H} = \left\{ h \in W^{1,2}(\mathbb{R}^N, e^{-F_{\gamma,N}(u)/\varepsilon} du) \,|\, \forall z \,, h(z) \in [0, 1] \,, h_{|A} = 1 \,, h_{|D} = 0 \right\}, \tag{2.8}$$

and the Dirichlet form Φ is given, for $h \in \mathcal{H}$, as

$$\Phi(h) = \varepsilon \int_{(A \cup D)^c} e^{-F_{\gamma,N}(u)/\varepsilon} \|\nabla h(u)\|_2^2 du. \tag{2.9}$$

Remark. Formula (2.6) gives an average of the mean transition time with respect to the equilibrium measure, that we will extensively use in what follows. A way to obtain the quantity $\mathbb{E}_z[\tau_D]$ consists in using Hölder and Harnack estimates [12] (as developed in Corollary 2.3)[5], but it is far from obvious whether this can be extended to give estimates that are uniform in N.

Formula (2.6) highlights the two terms for which we will prove uniform estimates: the capacity (Proposition 4.3) and the mass of $h_{A,D}$ (Proposition 4.9).

2.2. **Description of the Potential.** Let us describe in detail the potential $F_{\gamma,N}$, its stationary points, and in particular the minima and the 1-saddle points, through which the transitions occur.

The coupling strength γ specifies the geometry of $F_{\gamma,N}$. For instance, if we set $\gamma = 0$, we get a set of N bistable independent particles, thus the stationary points are

$$x^* = (\xi_1, \dots, \xi_N) \quad \forall i \in [\![1, N]\!],\ \xi_i \in \{-1, 0, 1\}. \tag{2.10}$$

To characterize their stability, we have to look to their Hessian matrix whose signs of the eigenvalues give us the index saddle of the point. It can be easily shown that, for $\gamma = 0$, the minima are those of the form (2.10) with no zero coordinates and the 1-saddle points have just one zero coordinate. As γ increases, the structure of the potential evolves and the number of stationary points decreases from 3^N to 3. We notice that, for all γ, the points

$$I_\pm = \pm(1, 1, \cdots, 1) \quad O = (0, 0, \cdots, 0) \tag{2.11}$$

are stationary, furthermore $I_\pm$ are minima. If we calculate the Hessian at the point O, we have

$$\nabla^2 F_{\gamma,N}(O) = \begin{pmatrix} -1+\gamma & -\frac{\gamma}{2} & 0 & \cdots & 0 & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & -1+\gamma & -\frac{\gamma}{2} & & & 0 \\ 0 & -\frac{\gamma}{2} & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & \ddots & \ddots & -\frac{\gamma}{2} \\ -\frac{\gamma}{2} & 0 & \cdots & 0 & -\frac{\gamma}{2} & -1+\gamma \end{pmatrix}, \tag{2.12}$$

whose eigenvalues are, for all $\gamma > 0$ and for $0 \le k \le N-1$,

$$\lambda_{k,N} = -\left(1 - 2\gamma \sin^2\left(\frac{k\pi}{N}\right)\right). \tag{2.13}$$

Set, for $k \ge 1$, $\gamma_k^N = \frac{1}{2\sin^2(k\pi/N)}$. Then these eigenvalues can be written in the form

$$\begin{cases} \lambda_{k,N} & = \lambda_{N-k,N} = -1 + \frac{\gamma}{\gamma_k^N},\ 1 \le k \le N-1 \\ \lambda_{0,N} & = \lambda_0 = -1. \end{cases} \tag{2.14}$$

Note that $(\gamma_k^N)_{k=1}^{\lfloor N/2 \rfloor}$ is a decreasing sequence, and so as γ increases, the number of non-positive eigenvalues $(\lambda_{k,N})_{k=0}^{N-1}$ decreases. When $\gamma > \gamma_1^N$, the only negative eigenvalue is -1. Thus

$$\gamma_1^N = \frac{1}{2\sin^2(\pi/N)} \tag{2.15}$$

is the threshold of the synchronization regime.

Lemma 2.1 (Synchronization Regime). *If $\gamma > \gamma_1^N$, the only stationary points of $F_{\gamma,N}$ are $I_\pm$ and O. $I_\pm$ are minima, O is a 1-saddle.*

This lemma was proven in [2] by using a Lyapunov function. This configuration is called the synchronization regime because the coupling between the particles is so strong that they all pass simultaneously through their respective saddle points in a transition between the stable equilibria ($I_\pm$).

In this paper, we will focus on this regime.

2.3. **Results for fixed** N. Let $\rho > 0$ and set $B_\pm \equiv B_\rho(I_\pm)$, where $B_\rho(x)$ denotes the ball of radius ρ centered at x. Equation (2.6) gives, with $A = B_-$ and $D = B_+$,

$$\mathbb{E}_{\nu_{B_-,B_+}}[\tau_{B_+}] = \frac{\int_{B_+^c} h_{B_-,B_+}(y) e^{-F_{\gamma,N}(y)/\varepsilon} dy}{\mathrm{cap}(B_-, B_+)}. \tag{2.16}$$

First, we obtain a sharp estimate for this transition time for fixed N:

Theorem 2.2. *Let $N > 2$ be given. For $\gamma > \gamma_1^N = \frac{1}{2\sin^2(\pi/N)}$, let $\sqrt{N} > \rho \geq \epsilon > 0$. Then*

$$\mathbb{E}_{\nu_{B_-,B_+}}[\tau_{B_+}] = 2\pi c_N e^{\frac{N}{4\epsilon}} (1 + O(\sqrt{\varepsilon |\ln \varepsilon|^3})) \tag{2.17}$$

with

$$c_N = \left[1 - \frac{3}{2+2\gamma}\right]^{\frac{e(N)}{2}} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \left[1 - \frac{3}{2 + \frac{\gamma}{\gamma_k^N}}\right] \tag{2.18}$$

where $e(N) = 1$ if N is even and 0 if N is odd.

Remark. The power 3 at $\ln \varepsilon$ is missing in [5] by mistake.

Remark. As mentioned above, for any fixed dimension, we can replace the probability measure ν_{B_-,B_+} by the Dirac measure on the single point I_-, using Hölder and Harnack inequalities [5]. This gives the following corollary:

Corollary 2.3. *Under the assumptions of Theorem 2.2, there exists $\alpha > 0$ such that*

$$\mathbb{E}_{I_-}[\tau_{B_+}] = 2\pi c_N e^{\frac{N}{4\epsilon}} (1 + O(\sqrt{\varepsilon |\ln \varepsilon|^3}). \tag{2.19}$$

Proof of the theorem. We apply Theorem 3.2 in [5]. For $\gamma > \gamma_1^N = \frac{1}{2\sin^2(\pi/N)}$, let us recall that there are only three stationary points: two minima $I_\pm$ and one saddle point O. One easily checks that $F_{\gamma,N}$ satisfies the following assumptions:

- $F_{\gamma,N}$ is polynomial in the $(x_i)_{i\in\Lambda}$ and so clearly C^3 on $\mathbb{R}^N$.
- $F_{\gamma,N}(x) \geq \frac{1}{4}\sum_{i\in\Lambda} x_i^4$ so $F_{\gamma,N} \underset{x\to\infty}{\longrightarrow} +\infty$.
- $\|\nabla F_{\gamma,N}(x)\|_2 \sim \|x\|_3^3$ as $\|x\|_2 \to \infty$.
- As $\Delta F_{\gamma,N}(x) \sim 3\|x\|_2^2$ ($\|x\|_2 \to \infty$), then $\|\nabla F_{\gamma,N}(x)\| - 2\Delta F_{\gamma,N}(x) \sim \|x\|_3^3$.

The Hessian matrix at the minima $I_\pm$ has the form

$$\nabla^2 F_{\gamma,N}(I_\pm) = \nabla^2 F_{\gamma,N}(O) + 3\mathrm{Id}, \tag{2.20}$$

whose eigenvalues are simply

$$\nu_{k,N} = \lambda_{k,N} + 3. \tag{2.21}$$

Then Theorem 3.1 of [5] can be applied and yields, for $\sqrt{N} > \rho > \epsilon > 0$, (recall the the negative eigenvalue of the Hessian at O is -1)

$$\mathbb{E}_{\nu_{B_-,B_+}}[\tau_{B_+}] = \frac{2\pi e^{\frac{N}{4\epsilon}} \sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}}{\sqrt{\det(\nabla^2 F_{\gamma,N}(I_-))}} (1 + O(\sqrt{\epsilon}|\ln \epsilon|^3)). \tag{2.22}$$

Finally, (2.14) and (2.21) give:

$$\det(\nabla^2 F_{\gamma,N}(I_-)) = \prod_{k=0}^{N-1} \nu_{k,N} = 2\nu_{N/2,N}^{e(N)} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \nu_{k,N}^2 = 2^N (1+\gamma)^{e(N)} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \left(1 + \frac{\gamma}{2\gamma_k^N}\right)^2 \tag{2.23}$$

$$|\det(\nabla^2 F_{\gamma,N}(O))| = \prod_{k=0}^{N-1} \lambda_{k,N} = \lambda_{N/2,N}^{e(N)} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \lambda_{k,N}^2 = (2\gamma - 1)^{e(N)} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \left(1 - \frac{\gamma}{\gamma_k^N}\right)^2. \tag{2.24}$$

Then,

$$c_N = \frac{\sqrt{\det(\nabla^2 F_{\gamma,N}(I_-))}}{\sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}} = \left[1 - \frac{3}{2+2\gamma}\right]^{\frac{e(N)}{2}} \prod_{k=1}^{\lfloor \frac{N-1}{2} \rfloor} \left[1 - \frac{3}{2 + \frac{\gamma}{\gamma_k^N}}\right] \tag{2.25}$$

and Theorem 2.2 is proved. □

Let us point out that the use of these estimates is a major obstacle to obtain a mean transition time starting from a single stable point with uniform error terms. That is the reason why we have introduced the equilibrium probability. However, there are still several difficulties to be overcome if we want to pass to the limit $N \uparrow \infty$.

(i) We must show that the prefactor c_N has a limit as $N \uparrow \infty$.
(ii) The exponential term in the hitting time tends to infinity with N. This suggests that one needs to rescale the potential $F_{\gamma,N}$ by a factor $1/N$, or equivalently, to increase the noise strength by a factor N.
(iii) One will need uniform control of error estimates in N to be able to infer the metastable behavior of the infinite dimensional system. This will be the most subtle of the problems involved.

3. Large N limit

As mentioned above, in order to prove a limiting result as N tends to infinity, we need to rescale the potential to eliminate the N-dependence in the exponential. Thus henceforth we replace $F_{\gamma,N}(x)$ by

$$G_{\gamma,N}(x) = N^{-1} F_{\gamma,N}(x). \tag{3.1}$$

This choice actually has a very nice side effect. Namely, as we always want to be in the regime where $\gamma \sim \gamma_1^N \sim N^2$, it is natural to parametrize the coupling constant with a fixed $\mu > 1$ as

$$\gamma^N = \mu \gamma_1^N = \frac{\mu}{2\sin^2(\frac{\pi}{N})} = \frac{\mu N^2}{2\pi^2}(1 + o(1)). \tag{3.2}$$

Then, if we replace the lattice by a lattice of spacing $1/N$, i.e. $(x_i)_{i\in\Lambda}$ is the discretization of a real function x on $[0,1]$ ($x_i = x(i/N)$), the resulting potential converges formally to

$$G_{\gamma^N,N}(x) \underset{N\to\infty}{\to} \int_0^1 \left(\frac{1}{4}[x(s)]^4 - \frac{1}{2}[x(s)]^2\right) ds + \frac{\mu}{4\pi^2} \int_0^1 \frac{[x'(s)]^2}{2} ds, \tag{3.3}$$

with $x(0) = x(1)$.

In the Euclidean norm, we have $\|I_\pm\|_2 = \sqrt{N}$, which suggests to rescale the size of neighborhoods. We consider, for $\rho > 0$, the neighborhoods $B_\pm^N = B_{\rho\sqrt{N}}(I_\pm)$. The volume $V(B_-^N) = V(B_+^N)$ goes to 0 if and only if $\rho < 1/2\pi e$, so given such a ρ, the balls $B_\pm^N$ are not as large as one might think. Let us also observe that

$$\frac{1}{\sqrt{N}}\|x\|_2 \underset{N\to\infty}{\longrightarrow} \|x\|_{L^2[0,1]} = \int_0^1 |x(s)|^2 ds. \tag{3.4}$$

Therefore, if $x \in B_+^N$ for all N, we get in the limit, $\|x-1\|_{L^2[0,1]} < \rho$.

The main result of this paper is the following uniform version of Theorem 2.2 with a rescaled potential $G_{\gamma,N}$.

Theorem 3.1. *Let $\mu \in]1,\infty[$, then there exists a constant, A, such that for all $N \geq 2$ and all $\varepsilon > 0$,*

$$\frac{1}{N}\mathbb{E}_{\nu_{B_-^N,B_+^N}}[\tau_{B_+^N}] = 2\pi c_N e^{1/4\varepsilon}(1 + R(\varepsilon, N)), \tag{3.5}$$

where c_N is defined in Theorem 2.2 and $|R(\varepsilon, N)| \leq A\sqrt{\varepsilon|\ln\varepsilon|^3}$. In particular,

$$\lim_{\varepsilon\downarrow 0}\lim_{N\uparrow\infty}\frac{1}{N}e^{-1/4\varepsilon}\mathbb{E}_{\nu_{B_-^N,B_+^N}}[\tau_{B_+^N}] = 2\pi V(\mu) \tag{3.6}$$

where

$$V(\mu) = \prod_{k=1}^{+\infty}\left[\frac{\mu k^2 - 1}{\mu k^2 + 2}\right] < \infty. \tag{3.7}$$

Remark. The appearance of the factor $1/N$ may at first glance seem disturbing. It corresponds however to the appropriate time rescaling when scaling the spatial coordinates i to i/N in order to recover the pde limit.

The proof of this theorem will be decomposed in two parts:

- convergence of the sequence c_N (Proposition 3.2);
- uniform control of the denominator (Proposition 4.3) and the numerator (Proposition 4.9) of Formula (2.16).

Convergence of the prefactor c_N. Our first step will be to control the behavior of c_N as $N \uparrow \infty$. We prove the following:

Proposition 3.2. *The sequence c_N converges: for $\mu > 1$, we set $\gamma = \mu\gamma_1^N$, then*

$$\lim_{N\uparrow\infty} c_N = V(\mu), \tag{3.8}$$

with $V(\mu)$ defined in (3.7).

Remark. This proposition immediately leads to

Corollary 3.3. *For $\mu \in]1,\infty[$, we set $\gamma = \mu\gamma_1^N$, then*

$$\lim_{N\uparrow\infty}\lim_{\varepsilon\downarrow 0}\frac{e^{-\frac{1}{4\varepsilon}}}{N}\mathbb{E}_{\nu_{B_-^N,B_+^N}}[\tau_{B_+^N}] = 2\pi V(\mu). \tag{3.9}$$

Of course such a result is unsatisfactory, since it does not tell us anything about a large system with specified fixed noise strength. To be able to interchange the limits regarding ε and N, we need a uniform control on the error terms.

Proof of the proposition. The rescaling of the potential introduces a factor $\frac{1}{N}$ for the eigenvalues, so that (2.22) becomes

$$\begin{aligned}\mathbb{E}_{\nu_{B_-^N,B_+^N}}[\tau_{B_+^N}] &= \frac{2\pi e^{\frac{1}{4\epsilon}} N^{-N/2+1}\sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}}{N^{-N/2}\sqrt{\det(\nabla^2 F_{\gamma,N}(I_-))}}(1+O(\sqrt{\epsilon}|\ln\epsilon|^3)) \\ &= 2\pi N c_N e^{\frac{1}{4\epsilon}}(1+O(\sqrt{\epsilon}|\ln\epsilon|^3)). \end{aligned} \tag{3.10}$$

Then, with $u_k^N = \frac{3}{2+\mu\frac{\gamma_1^N}{\gamma_k^N}}$,

$$c_N = \left[1-\frac{3}{2+2\mu\gamma_1^N}\right]^{\frac{e(N)}{2}} \prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\left[1-u_k^N\right]. \tag{3.11}$$

To prove the convergence, let us consider the $(\gamma_k^N)_{k=1}^{N-1}$. For all $k\geq 1$, we have

$$\frac{\gamma_1^N}{\gamma_k^N} = \frac{\sin^2(\frac{k\pi}{N})}{\sin^2(\frac{\pi}{N})} = k^2 + (1-k^2)\frac{\pi^2}{3N^2} + o\left(\frac{1}{N^2}\right). \tag{3.12}$$

Hence, $u_k^N \underset{N\to+\infty}{\longrightarrow} v_k = \frac{3}{2+\mu k^2}$. Thus, we want to show that

$$c_N \underset{N\to+\infty}{\longrightarrow} \prod_{k=1}^{+\infty}(1-v_k) = V(\mu). \tag{3.13}$$

Using that, for $0\leq t\leq\frac{\pi}{2}$,

$$0 < t^2(1-\frac{t^2}{3}) \leq \sin^2(t) \leq t^2, \tag{3.14}$$

we get the following estimates for $\frac{\gamma_1^N}{\gamma_k^N}$: set $a=\left(1-\frac{\pi^2}{12}\right)$, for $1\leq k\leq N/2$,

$$ak^2 = \left(1-\frac{\pi^2}{12}\right)k^2 \leq k^2\left(1-\frac{k^2\pi^2}{3N^2}\right) \leq \frac{\gamma_1^N}{\gamma_k^N} = \frac{\sin^2(\frac{k\pi}{N})}{\sin^2(\frac{\pi}{N})} \leq \frac{k^2}{1-\frac{\pi^2}{3N^2}}. \tag{3.15}$$

Then, for $N\geq 2$ and for all $1\leq k\leq N/2$,

$$-\frac{k^4\pi^2}{3N^2} \leq \frac{\gamma_1^N}{\gamma_k^N} - k^2 \leq \frac{k^2\pi^2}{3N^2\left(1-\frac{\pi^2}{3N^2}\right)} \leq \frac{k^2\pi^2}{N^2}. \tag{3.16}$$

Let us introduce

$$V_m = \prod_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}(1-v_k), \quad U_{N,m} = \prod_{k=1}^{\lfloor\frac{m-1}{2}\rfloor}\left(1-u_k^N\right). \tag{3.17}$$

Then

$$\left|\ln\frac{U_{N,N}}{V_N}\right| = \left|\ln\prod_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\frac{1-u_k^N}{1-v_k}\right| \leq \sum_{k=1}^{\lfloor\frac{N-1}{2}\rfloor}\left|\ln\frac{1-u_k^N}{1-v_k}\right|. \tag{3.18}$$

Using (3.15) and (3.16), we obtain, for all $1\leq k\leq N/2$,

$$\left|\frac{v_k-u_k^N}{1-v_k}\right| = \frac{3\mu\left|\frac{\gamma_1^N}{\gamma_k^N}-k^2\right|}{(-1+\mu k^2)\left(2+\mu\frac{\gamma_1^N}{\gamma_k^N}\right)} \leq \frac{\mu k^4\pi^2}{N^2(-1+\mu k^2)(2+\mu a k^2)} \leq \frac{C}{N^2} \tag{3.19}$$

with C a constant independent of k. Therefore, for $N > N_0$,

$$\left| \ln \frac{1-u_k^N}{1-v_k} \right| = \left| \ln \left(1 + \frac{v_k - u_k^N}{1-v_k} \right) \right| \le \frac{C'}{N^2}. \tag{3.20}$$

Hence

$$\left| \ln \frac{U_{N,N}}{V_N} \right| \le \frac{C'}{N} \underset{N\to+\infty}{\longrightarrow} 0. \tag{3.21}$$

As $\sum |v_k| < +\infty$, we get $\lim_{N\to+\infty} V_N = V(\mu) > 0$, and thus (3.13) is proved. $\square$

4. Estimates on capacities

To prove Theorem 3.1, we prove uniform estimates of the denominator and numerator of (2.6), namely the capacity and the mass of the equilibrium potential.

4.1. **Uniform control in large dimensions for capacities.** A crucial step is the control of the capacity. This will be done with the help of the Dirichlet principle (2.7). We will obtain the asymptotics by using a Laplace-like method. The exponential factor in the integral (2.9) is largely predominant at the points where h is likely to vary the most, that is around the saddle point O. Therefore we need some good estimates of the potential near O.

4.1.1. *Local Taylor approximation.* This subsection is devoted to the quadratic approximations of the potential which are quite subtle. We will make a change of basis in the neighborhood of the saddle point O that will diagonalize the quadratic part.

Recall that the potential $G_{\gamma,N}$ is of the form

$$G_{\gamma,N}(x) = -\frac{1}{2N}(x, [\mathrm{Id} - \mathbb{D}]x) + \frac{1}{4N}\|x\|_4^4. \tag{4.1}$$

where the operator $\mathbb{D}$ is given by $\mathbb{D} = \gamma \left[\mathrm{Id} - \frac{1}{2}(\Sigma + \Sigma^*)\right]$ and $(\Sigma x)_j = x_{j+1}$. The linear operator $(\mathrm{Id} - \mathbb{D}) = -\nabla^2 F_{\gamma,N}(O)$ has eigenvalues $-\lambda_{k,N}$ and eigenvectors $v_{k,N}$ with components $v_{k,N}(j) = \omega^{jk}$, with $\omega = e^{i2\pi/N}$.
Let us change coordinates by setting

$$\hat{x}_j = \sum_{k=0}^{N-1} \omega^{-jk} x_k. \tag{4.2}$$

Then the inverse transformation is given by

$$x_k = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{jk} \hat{x}_j = x_k(\hat{x}). \tag{4.3}$$

Note that the map $x \to \hat{x}$ maps $\mathbb{R}^N$ to the set

$$\widehat{\mathbb{R}}^N = \left\{ \hat{x} \in \mathbb{C}^N : \hat{x}_k = \overline{\hat{x}_{N-k}} \right\} \tag{4.4}$$

endowed with the standard inner product on $\mathbb{C}^N$.
Notice that, expressed in terms of the variables $\hat{x}$, the potential (1.2) takes the form

$$G_{\gamma,N}(x(\hat{x})) = \frac{1}{2N^2} \sum_{k=0}^{N-1} \lambda_{k,N} |\hat{x}_k|^2 + \frac{1}{4N}\|x(\hat{x})\|_4^4. \tag{4.5}$$

Our main concern will be the control of the non-quadratic term in the new coordinates. To that end, we introduce the following norms on Fourier space:

$$\|\hat{x}\|_{p,\mathcal{F}} = \left(\frac{1}{N} \sum_{i=0}^{N-1} |\hat{x}|^p \right)^{1/p} = \frac{1}{N^{1/p}} \|\hat{x}\|_p. \tag{4.6}$$

The factor $1/N$ is the suitable choice to make the map $x \to \hat{x}$ a bounded map between L^p spaces. This implies that the following estimates hold (see [16], Vol. 1, Theorem IX.8):

Lemma 4.1. *With the norms defined above, we have*

(i) *the Parseval identity,*

$$\|x\|_2 = \|\hat{x}\|_{2,\mathcal{F}}, \tag{4.7}$$

and

(ii) *the Hausdorff-Young inequalities: for $1 \le q \le 2$ and $p^{-1} + q^{-1} = 1$, there exists a finite, N-independent constant C_q such that*

$$\|x\|_p \le C_q \|\hat{x}\|_{q,\mathcal{F}}. \tag{4.8}$$

In particular

$$\|x\|_4 \le C_{4/3} \|\hat{x}\|_{4/3,\mathcal{F}}. \tag{4.9}$$

Let us introduce the change of variables, defined by the complex vector z, as

$$z = \frac{\hat{x}}{N}. \tag{4.10}$$

Let us remark that $z_0 = \frac{1}{N}\sum_{k=1}^{N-1} x_k \in \mathbb{R}$. In the variable z, the potential takes the form

$$\widetilde{G}_{\gamma,N}(z) = G_{\gamma,N}\left(x(Nz)\right) = \frac{1}{2}\sum_{k=0}^{N-1} \lambda_{k,N}|z_k|^2 + \frac{1}{4N}\|x(Nz)\|_4^4. \tag{4.11}$$

Moreover, by (4.7) and (4.10)

$$\|x(Nz)\|_2^2 = \|Nz\|_{2,\mathcal{F}}^2 = \frac{1}{N}\|Nz\|_2^2. \tag{4.12}$$

In the new coordinates the minima are now given by

$$I_\pm = \pm(1, 0, \ldots, 0). \tag{4.13}$$

In addition, $z(B_-^N) = z(B_{\rho\sqrt{N}}(I_-)) = B_\rho(I_-)$ where the last ball is in the new coordinates.

Lemma 4.1 will allow us to prove the following important estimates. For $\delta > 0$, we set

$$C_\delta = \left\{ z \in \widehat{\mathbb{R}}^N : |z_k| \le \delta \frac{r_{k,N}}{\sqrt{|\lambda_{k,N}|}},\ 0 \le k \le N-1 \right\}, \tag{4.14}$$

where $\lambda_{k,N}$ are the eigenvalues of the Hessian at O as given in (2.14) and $r_{k,N}$ are constants that will be specified below. Using (3.15), we have, for $3 \le k \le N/2$,

$$\lambda_{k,N} \ge k^2 \left(1 - \frac{\pi^2}{12}\right)\mu - 1. \tag{4.15}$$

Thus $(\lambda_{k,N})$ verifies $\lambda_{k,N} \ge ak^2$, for $1 \le k \le N/2$, with some a, independent of N.

The sequence $(r_{k,N})$ is constructed as follows. Choose an increasing sequence, $(\rho_k)_{k\geq 1}$, and set

$$\begin{cases} r_{0,N} &= 1 \\ r_{k,N} &= r_{N-k,N} = \rho_k, \quad 1 \leq k \leq \left\lfloor \frac{N}{2} \right\rfloor. \end{cases} \tag{4.16}$$

Let, for $p \geq 1$,

$$K_p = \left(\sum_{k\geq 1} \frac{\rho_k^p}{k^p} \right)^{1/p}. \tag{4.17}$$

Note that if K_{p_0} is finite then, for all $p_1 > p_0$, K_{p_1} is finite. With this notation we have the following key estimate.

Lemma 4.2. *For all $p \geq 2$, there exist finite constants B_p, such that, for $z \in C_\delta$,*

$$\|x(Nz)\|_p^p \leq \delta^p N B_p \tag{4.18}$$

if K_q is finite, with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The Hausdorff-Young inequality (Lemma 4.1) gives us:

$$\|x(Nz)\|_p \leq C_q \|Nz\|_{q,\mathcal{F}}. \tag{4.19}$$

Since $z \in C_\delta$, we get

$$\|Nz\|_{q,\mathcal{F}}^q \leq \delta^q N^{q-1} \sum_{k=0}^{N-1} \frac{r_{k,N}^q}{\lambda_k^{q/2}}. \tag{4.20}$$

Then

$$\sum_{k=0}^{N-1} \frac{r_{k,N}^q}{\lambda_k^{q/2}} = \frac{1}{\lambda_0^{q/2}} + 2 \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{r_{k,N}^q}{\lambda_k^{q/2}} \leq \frac{1}{\lambda_0^{q/2}} + \frac{2}{a^{q/2}} \sum_{k=1}^{\lfloor N/2 \rfloor} \frac{\rho_k^q}{k^q} \leq \frac{1}{\lambda_0^{q/2}} + \frac{2}{a^{q/2}} K_q^q = D_q^q \tag{4.21}$$

which is finite if K_q is finite. Therefore,

$$\|x(Nz)\|_p^p \leq \delta^p N^{(q-1)\frac{p}{q}} C_q^p D_q^p, \tag{4.22}$$

which gives us the result since $(q-1)\frac{p}{q} = 1$. □

We have all what we need to estimate the capacity.

4.1.2. *Capacity Estimates.* Let us now prove our main theorem.

Proposition 4.3. *There exists a constant A, such that, for all $\varepsilon < \varepsilon_0$ and for all N,*

$$\frac{\operatorname{cap}\left(B_+^N, B_-^N\right)}{N^{N/2-1}} = \varepsilon \sqrt{2\pi\varepsilon}^{N-2} \frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}} \left(1 + R(\varepsilon, N)\right), \tag{4.23}$$

where $|R(\varepsilon, N)| \leq A\sqrt{\varepsilon |\ln \varepsilon|^3}$.

The proof will be decomposed into two lemmata, one for the upper bound and the other for the lower bound. The proofs are quite different but follow the same idea. We have to estimate some integrals. We isolate a neighborhood around the point O of interest. We get an approximation of the potential on this neighborhood, we bound the remainder and we estimate the integral on the suitable neighborhood.

In what follows, constants independent of N are denoted A_i.

Upper bound. The first lemma we prove is the upper bound for Proposition 4.3.

Lemma 4.4. *There exists a constant A_0 such that for all ε and for all N,*

$$\frac{\mathrm{cap}\left(B_+^N, B_-^N\right)}{N^{N/2-1}} \leq \varepsilon\sqrt{2\pi\varepsilon}^{N-2}\frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}}\left(1 + A_0\varepsilon|\ln\varepsilon|^2\right). \tag{4.24}$$

Proof. This lemma is proved in [5] in the finite dimension setting. We use the same strategy, but here we take care to control the integrals appearing uniformly in the dimension.

We will denote the quadratic approximation of $\widetilde{G}_{\gamma,N}$ by F_0, i.e.

$$F_0(z) = \sum_{k=0}^{N-1}\frac{\lambda_{k,N}|z_k|^2}{2} = -\frac{z_0^2}{2} + \sum_{k=1}^{N-1}\frac{\lambda_{k,N}|z_k|^2}{2}. \tag{4.25}$$

On C_δ, we can control the non-quadratic part through Lemma 4.2.

Lemma 4.5. *There exists a constant A_1 and δ_0, such that for all N, $\delta < \delta_0$ and all $z \in C_\delta$,*

$$\left|\widetilde{G}_{\gamma,N}(z) - F_0(z)\right| \leq A_1\delta^4. \tag{4.26}$$

Proof. Using (4.11), we see that

$$\widetilde{G}_{\gamma,N}(z) - F_0(z) = \frac{1}{4N}\|x(Nz)\|_4^4. \tag{4.27}$$

We choose a sequence $(\rho_k)_{k\geq 1}$ such that $K_{4/3}$ is finite.

Thus, it follows from Lemma 4.2, with $A_1 = \frac{1}{4}B_4$, that

$$\left|\widetilde{G}_{\gamma,N}(z) - \frac{1}{2}\sum_{k=0}^{N-1}\lambda_{k,N}|z_k|^2\right| \leq A_1\delta^4, \tag{4.28}$$

as desired. □

We obtain the upper bound of Lemma 4.4 by choosing a test function h^+. We change coordinates from x to z as explained in (4.10). A simple calculation shows that

$$\|\nabla h(x)\|_2^2 = N^{-1}\|\nabla\tilde{h}(z)\|_2^2, \tag{4.29}$$

where $\tilde{h}(z) = h(x(z))$ under our coordinate change.

For δ sufficiently small, we can ensure that, for $z \notin C_\delta$ with $|z_0| \leq \delta$,

$$\widetilde{G}_{\gamma,N}(z) \geq F_0(z) = -\frac{z_0^2}{2} + \frac{1}{2}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2 \geq -\frac{\delta^2}{2} + 2\delta^2 \geq \delta^2. \tag{4.30}$$

Therefore, the strip

$$S_\delta \equiv \{x|\, x = x(Nz),\ |z_0| < \delta\} \tag{4.31}$$

separates $\mathbb{R}^N$ into two disjoint sets, one containing I_- and the other one containing I_+, and for $x \in S_d \setminus C_\delta$, $G_{\gamma,N}(x) \geq \delta^2$.

The complement of S_δ consists of two connected components Γ_+, Γ_- which contain I_+ and I_-, respectively. We define

$$\tilde{h}^+(z) = \begin{cases} 1 & \text{for } z \in \Gamma_- \\ 0 & \text{for } z \in \Gamma_+ \\ f(z_0) & \text{for } z \in C_\delta \\ \text{arbitrary} & \text{on } S_\delta \setminus C_\delta \ \text{ but } \|\nabla h^+\|_2 \leq \frac{c}{\delta}. \end{cases}, \tag{4.32}$$

where f satisfies $f(\delta) = 0$ and $f(-\delta) = 1$ and will be specified later.

Taking into account the change of coordinates, the Dirichlet form (2.9) evaluated on h^+ provides the upper bound

$$\begin{aligned} \Phi(h^+) &= N^{N/2-1}\varepsilon \int_{z((B_-^N \cup B_+^N)^c)} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} \|\nabla \tilde{h}^+(z)\|_2^2 dz \qquad (4.33) \\ &\leq N^{N/2-1}\left[\varepsilon \int_{C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} \left(f'(z_0)\right)^2 dz + \varepsilon\delta^{-2}c^2 \int_{S_\delta \setminus C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz\right]. \end{aligned}$$

The first term will give the dominant contribution. Let us focus on it first. We replace $\widetilde{G}_{\gamma,N}$ by F_0, using the bound (4.26), and for suitably chosen δ, we obtain

$$\begin{aligned} \int_{C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} \left(f'(z_0)\right)^2 dz &\leq \left(1 + 2A_1 \frac{\delta^4}{\varepsilon}\right) \int_{C_\delta} e^{-F_0(z)/\varepsilon} \left(f'(z_0)\right)^2 dz \\ &= \left(1 + 2A_1 \frac{\delta^4}{\varepsilon}\right) \int_{D_\delta} e^{-\frac{1}{2\varepsilon}\sum_{k=1}^{N-1} \lambda_{k,N}|z_k|^2} dz_1 \dots dz_{N-1} \\ &\quad \times \int_{-\delta}^{\delta} \left(f'(z_0)\right)^2 e^{z_0^2/2\varepsilon} dz_0. \end{aligned} \tag{4.34}$$

Here we have used that we can write C_δ in the form $[-\delta, \delta] \times D_\delta$. As we want to calculate an infimum, we choose a function f which minimizes the integral $\int_{-\delta}^{\delta} \left(f'(z_0)\right)^2 e^{z_0^2/2\varepsilon} dz_0$. A simple computation leads to the choice

$$f(z_0) = \frac{\int_{z_0}^{\delta} e^{-t^2/2\varepsilon} dt}{\int_{-\delta}^{\delta} e^{-t^2/2\varepsilon} dt}. \tag{4.35}$$

Therefore

$$\int_{C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} \left(f'(z_0)\right)^2 dz \leq \frac{\int_{C_\delta} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1} |\lambda_{k,N}||z_k|^2} dz}{\left(\int_{-\delta}^{\delta} e^{-\frac{1}{2\varepsilon} z_0^2} dz_0\right)^2} \left(1 + 2A_1 \frac{\delta^4}{\varepsilon}\right). \tag{4.36}$$

Choosing $\delta = \sqrt{K\varepsilon|\ln\varepsilon|}$, a simple calculation shows that there exists A_2 such that

$$\frac{\int_{C_\delta} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1} |\lambda_{k,N}||z_k|^2} dz}{\left(\int_{-\delta}^{\delta} e^{-\frac{1}{2} z_0^2/\varepsilon} dz_0\right)^2} \leq \sqrt{2\pi\varepsilon}^{N-2} \frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}} (1 + A_2\varepsilon). \tag{4.37}$$

The second term in (4.33) is bounded above in the following lemma.

Lemma 4.6. *For $\delta = \sqrt{K\varepsilon|\ln(\varepsilon)|}$ and $\rho_k = 4k^\alpha$, with $0 < \alpha < 1/4$, there exists $A_3 < \infty$, such that for all N and $0 < \varepsilon < 1$,*

$$\int_{S_\delta \setminus C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz \leq \frac{A_3 \sqrt{2\pi\varepsilon}^{N-2}}{\sqrt{|\det\left(\nabla^2 F_{\gamma,N}(O)\right)|}} \varepsilon^{3K/2+1}. \tag{4.38}$$

Proof. Clearly, by (4.11),

$$\widetilde{G}_{\gamma,N}(z) \geq -\frac{z_0^2}{2} + \frac{1}{2}\sum_{k=1}^{N-1} \lambda_{k,N}|z_k|^2. \tag{4.39}$$

Thus

$$\begin{aligned}
&\int_{S_\delta \setminus C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz \\
\leq\ & \int_{-\delta}^{\delta} dz_0 \int_{\exists_{k=1}^{N-1}: |z_k| \geq \delta r_{k,N}/\sqrt{\lambda_{k,N}}} dz_1 \ldots dz_{N-1} e^{-\frac{1}{2\varepsilon}\sum_{k=0}^{N-1} \lambda_{k,N}|z_k|^2} \\
\leq\ & \int_{-\delta}^{\delta} e^{+z_0^2/2\varepsilon} dz_0 \sum_{k=1}^{N-1} \int_{|z_k| \geq \delta r_{k,N}/\sqrt{\lambda_{k,N}}} e^{-\lambda_{k,N}|z_k|^2/2\varepsilon} dz_k \\
& \quad \times \prod_{1 \leq i \neq k \leq N-1} \int_{\mathbb{R}} e^{-\lambda_{i,N}|z_i|^2/2\varepsilon} dz_i \\
\leq\ & 2e^{\delta^2/2\varepsilon} \sqrt{\frac{2\varepsilon}{\pi}} \sqrt{\prod_{i=1}^{N-1} 2\pi\varepsilon \lambda_{i,N}^{-1}} \sum_{k=1}^{N-1} r_{k,n}^{-1} e^{-\delta^2 r_{k,N}^2/2\varepsilon}.
\end{aligned} \tag{4.40}$$

Now,

$$\begin{aligned}
\sum_{k=1}^{N-1} r_{k,N}^{-1} e^{-\delta^2 r_{k,N}^2/2\varepsilon} &= \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} r_{k,N}^{-1} e^{-\delta^2 r_{k,N}^2/2\varepsilon} + \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} r_{N-k,N}^{-1} e^{-\delta^2 r_{N-k,N}^2/2\varepsilon} \\
&\leq 2\sum_{k=1}^{\infty} \rho_k^{-1} e^{-\delta^2 \rho_k^2/2\varepsilon}.
\end{aligned} \tag{4.41}$$

We choose $\rho_k = 4k^\alpha$ with $0 < \alpha < 1/4$ to ensure that $K_{4/3}$ is finite. With our choice for δ, the sum in (4.41) is then given by

$$\frac{1}{4}\sum_{n=1}^{\infty} n^{-\alpha} \varepsilon^{8Kn^{2\alpha}} \leq \frac{1}{4}\varepsilon^{2K} \sum_{n=1}^{\infty} \varepsilon^{6Kn^{2\alpha}} \leq C\varepsilon^{2K}, \tag{4.42}$$

since the sum over n is clearly convergent. Putting all the parts together, we get that

$$\int_{S_\delta \setminus C_\delta} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz \leq C\varepsilon^{3K/2+1} \sqrt{2\pi\varepsilon}^{N-2} \frac{1}{\sqrt{|\det(\nabla^2 F_{\gamma,N}(O))|}} \tag{4.43}$$

and Lemma 4.6 is proven. □

Finally, using (4.36), (4.33), (4.37), and (4.38), we obtain the upper bound

$$\frac{\Phi(h^+)}{N^{N/2-1}} \leq \frac{\varepsilon\sqrt{2\pi\varepsilon}^{N-2}}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}} (1 + A_2\varepsilon) \left(1 + 2A_1\varepsilon|\ln\varepsilon|^2 + A_3'\varepsilon^{3K/2}\right) \tag{4.44}$$

with the choice $\rho_k = 4k^\alpha$, $0 < \alpha < 1/4$ and $\delta = \sqrt{K\varepsilon|\ln\varepsilon|}$. Note that all constants are independent of N. Thus Lemma 4.4 is proven. □

Lower Bound The idea here (as already used in [6]) is to get a lower bound by restricting the state space to a narrow corridor from I_- to I_+ that contains the relevant paths and along which the potential is well controlled. We will prove the following lemma.

Lemma 4.7. *There exists a constant $A_4 < \infty$ such that for all ε and for all N,*

$$\frac{\operatorname{cap}\left(B_+^N, B_-^N\right)}{N^{N/2-1}} \geq \varepsilon\sqrt{2\pi\varepsilon}^{N-2} \frac{1}{\sqrt{|\det(\nabla F_{\gamma,N}(0))|}} \left(1 - A_4\sqrt{\varepsilon|\ln\varepsilon|^3}\right). \tag{4.45}$$

Proof. Given a sequence, $(\rho_k)_{k\geq1}$, with $r_{k,N}$ defined as in (4.16), we set

$$\widehat{C}_\delta = \left\{z_0 \in]-1+\rho, 1-\rho[, |z_k| \leq \delta\, r_{k,N}/\sqrt{\lambda_{k,N}}\right\}. \tag{4.46}$$

The restriction $|z_0| < 1-\rho$ is made to ensure that $\widehat{C}_\delta$ is disjoint from $B_\pm$ since in the new coordinates (4.10) $I_\pm = \pm(1,0,\ldots,0)$.

Clearly, if h^* is the minimizer of the Dirichlet form, then

$$\operatorname{cap}\left(B_-^N, B_+^N\right) = \Phi(h^*) \geq \Phi_{\widehat{C}_\delta}(h^*), \tag{4.47}$$

where $\Phi_{\widehat{C}_\delta}$ is the Dirichlet form for the process on $\widehat{C}_\delta$,

$$\Phi_{\widehat{C}_\delta}(h) = \varepsilon\int_{\widehat{C}_\delta} e^{-G_{\gamma,N}(x)/\varepsilon}\|\nabla h(x)\|_2^2 dx = N^{N/2-1}\varepsilon\int_{z(\widehat{C}_\delta)} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon}\|\nabla\tilde{h}(z)\|_2^2 dz. \tag{4.48}$$

To get our lower bound we now use simply that

$$\|\nabla\tilde{h}(z)\|_2^2 = \sum_{k=0}^{N-1}\left|\frac{\partial\tilde{h}^*}{\partial z_k}\right|^2 \geq \left|\frac{\partial\tilde{h}^*}{\partial z_0}\right|^2, \tag{4.49}$$

so that

$$\frac{\Phi(h^*)}{N^{N/2-1}} \geq \varepsilon\int_{z(\widehat{C}_\delta)} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon}\left|\frac{\partial\tilde{h}^*}{\partial z_0}(z)\right|^2 dz = \widetilde{\Phi}_{\widehat{C}_\delta}(\tilde{h}^*) \geq \min_{h\in\mathcal{H}}\widetilde{\Phi}_{\widehat{C}_\delta}(\tilde{h}). \tag{4.50}$$

The remaining variational problem involves only functions depending on the single coordinate z_0, with the other coordinates, $z_\perp = (z_i)_{1\leq i\leq N-1}$, appearing only as parameters. The corresponding minimizer is readily found explicitly as

$$\tilde{h}^-(z_0, z_\perp) = \frac{\int_{z_0}^{1-\rho} e^{\widetilde{G}_{\gamma,N}(s,z_\perp)/\varepsilon} ds}{\int_{-1+\rho}^{1-\rho} e^{\widetilde{G}_{\gamma,N}(s,z_\perp)/\varepsilon} ds} \tag{4.51}$$

and hence the capacity is bounded from below by

$$\frac{\operatorname{cap}\left(B_-^N, B_+^N\right)}{N^{N/2-1}} \geq \widetilde{\Phi}_{\widehat{C}_\delta}(\tilde{h}^-) = \varepsilon\int_{\widehat{C}_\delta^\perp}\left(\int_{-1+\rho}^{1-\rho} e^{\widetilde{G}_{\gamma,N}(z_0,z_\perp)/\varepsilon} dz_0\right)^{-1} dz_\perp. \tag{4.52}$$

Next, we have to evaluate the integrals in the r.h.s. above. The next lemma provides a suitable approximation of the potential on $\widehat{C}_\delta$. Note that since z_0 is no longer small, we only expand in the coordinates $z_\perp$.

Lemma 4.8. *Let $r_{k,N}$ be chosen as before with $\rho_k = 4k^\alpha$, $0 < \alpha < 1/4$. Then there exists a constant, A_5, and $\delta_0 > 0$, such that, for all N and $\delta < \delta_0$, on $\widehat{C}_\delta$,*

$$\left|\widetilde{G}_{\gamma,N}(z) - \left(-\frac{1}{2}z_0^2 + \frac{1}{4}z_0^4 + \frac{1}{2}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2 + z_0^2 f(z_\perp)\right)\right| \leq A_5\delta^3, \tag{4.53}$$

where

$$f(z_\perp) \equiv \frac{3}{2}\sum_{k=1}^{N-1} |z_k|^2. \tag{4.54}$$

Proof. We analyze the non-quadratic part of the potential on $\widehat{C}_\delta$, using (4.11) and (4.3)

$$\frac{1}{N}\|x(Nz)\|_4^4 = \frac{1}{N}\sum_{i=0}^{N-1}|x_i(Nz)|^4 = \frac{1}{N}\sum_{i=0}^{N-1}\left|z_0+\sum_{k=1}^{N-1}\omega^{ik}z_k\right|^4 = \frac{z_0^4}{N}\sum_{i=0}^{N-1}|1+u_i|^4 \tag{4.55}$$

where $u_i = \frac{1}{z_0}\sum_{k=1}^{N-1}\omega^{ik}z_k$. Note that $\sum_{i=0}^{N-1}u_i = 0$ and $u = \frac{1}{z_0}x\left(N(0,z_\perp)\right)$ and that for $z \in \widehat{\mathbb{R}}^N$, u_i is real. Thus

$$\sum_{i=0}^{N-1}|1+u_i|^4 = N + \sum_{i=0}^{N-1}\left(6u_i^2+4u_i^3+u_i^4\right), \tag{4.56}$$

we get that

$$\left|\frac{1}{N}\|x(Nz)\|_4^4 - z_0^4\Big(1+\frac{6}{N}\sum_{i=0}^{N-1}u_i^2\Big)\right| \le \frac{z_0^4}{N}\left(4\|u\|_3^3+\|u\|_4^4\right). \tag{4.57}$$

A simple computation shows that

$$\frac{6}{N}\sum_i u_i^2 = \frac{6}{z_0^2}\sum_{k\neq 0}|z_k|^2. \tag{4.58}$$

Thus as $|z_0|\le 1$, we see that

$$\left|\frac{1}{N}\|x(Nz)\|_4^4 - z_0^4 - 6z_0^2\sum_{k\neq 0}|z_k|^2\right| \le \frac{1}{N}\big(4\|x(N(0,z_\perp))\|_3^3+\|x(N(0,z_\perp))\|_4^4\big). \tag{4.59}$$

Using again Lemma 4.2, we get

$$\begin{aligned}\|x(N(0,z_\perp))\|_3^3 &\le B_3N\delta^3 \\ \|x(N(0,z_\perp))\|_4^4 &\le B_4N\delta^4.\end{aligned} \tag{4.60}$$

Therefore, Lemma 4.8 is proved, with $A_5 = 4B_3 + B_4\delta_0$. □

We use Lemma 4.8 to obtain the upper bound

$$\int_{-1+\rho}^{1-\rho} e^{\widetilde{G}_{\gamma,N}(z_0,z_\perp)/\varepsilon}dz_0 \le \exp\left(\tfrac{1}{2\varepsilon}\textstyle\sum_{k\neq 0}\lambda_{k,N}|z_k|^2+\tfrac{A_5\delta^3}{\varepsilon}\right)g(z_\perp), \tag{4.61}$$

where

$$g(z_\perp) = \int_{-1+\rho}^{1-\rho}\exp\left(-\varepsilon^{-1}\left(\frac{1}{2}z_0^2-\frac{1}{4}z_0^4-z_0^2f(z_\perp)\right)\right)dz_0. \tag{4.62}$$

This integral is readily estimate via Laplace's method as

$$g(z_\perp) = \frac{\sqrt{2\pi\varepsilon}}{\sqrt{1-2f(z_\perp)}}\left(1+O(\varepsilon)\right) = \sqrt{2\pi\varepsilon}\left(1+O(\varepsilon)+O(\delta^2)\right). \tag{4.63}$$

Inserting this estimate into (4.52), it remains to carry out the integrals over the vertical coordinates which yields

$$\begin{aligned}\widetilde{\Phi}_{\widehat{C}_\delta}(\tilde{h}^-) &\geq \varepsilon\int_{\widehat{C}_\delta^\perp} \exp\left(-\frac{1}{2\varepsilon}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2 - \frac{A_5\delta^3}{\varepsilon}\right)\frac{1}{\sqrt{2\pi\varepsilon}}\left(1+O(\varepsilon)+O(\delta^2)\right)dz_\perp \\ &= \sqrt{\frac{\varepsilon}{2\pi}}\int_{\widehat{C}_\delta^\perp}\exp\left(-\frac{1}{2\varepsilon}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2\right)dz_\perp\left(1+O(\varepsilon)+O(\delta^2)+O(\delta^3/\varepsilon)\right).\end{aligned} \tag{4.64}$$

The integral is readily bounded by

$$\begin{aligned}&\int_{\widehat{C}_\delta^\perp}\exp\left(-\frac{1}{2\varepsilon}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2\right)dz_\perp \geq \int_{\mathbb{R}^{N-1}}\exp\left(-\frac{1}{2\varepsilon}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2\right)dz_\perp \\ &-\sum_{k=1}^{N-1}\int_{\mathbb{R}}dz_1\ldots\int_{|z_k|\geq\delta r_{k,N}/\sqrt{\lambda_{k,N}}}\ldots\int_{\mathbb{R}}dz_{N-1}\exp\left(-\frac{1}{2\varepsilon}\sum_{k=1}^{N-1}\lambda_{k,N}|z_k|^2\right) \\ &\geq \sqrt{2\pi\varepsilon}^{N-1}\prod_{i=1}^{N-1}\sqrt{\lambda_{i,N}}^{-1}\left(1-\sqrt{\frac{2\varepsilon}{\pi}}\delta^{-1}\sum_{k=1}^{N-1}r_{k,N}^{-1}e^{-\delta^2 r_{k,N}^2/2\varepsilon}\right) \\ &= \sqrt{2\pi\varepsilon}^{N-1}\frac{1}{\sqrt{|\det F_{\gamma,N}(O)|}}\left(1+O(\varepsilon^K)\right),\end{aligned} \tag{4.65}$$

when $\delta = \sqrt{K\varepsilon\ln\varepsilon}$ and $O(\varepsilon^K)$ uniform in N. Putting all estimates together, we arrive at the assertion of Lemma 4.7.

□

4.2. **Uniform estimate of the mass of the equilibrium potential.** We will prove the following proposition.

Proposition 4.9. *There exists a constant A_6 such that, for all $\varepsilon < \varepsilon_0$ and all N,*

$$\frac{1}{N^{N/2}}\int_{B_+^{N^c}} h^*_{B_-^N,B_+^N}(x)e^{-G_{\gamma,N}(x)/\varepsilon}dx = \frac{\sqrt{2\pi\varepsilon}^N\exp\left(\frac{1}{4\varepsilon}\right)}{\sqrt{\det(\nabla F_{\gamma,N}(I_-))}}\left(1+R(N,\varepsilon)\right), \tag{4.66}$$

where $|R(N,\varepsilon)| \leq A_6\sqrt{\varepsilon|\ln\varepsilon|^3}$.

Proof. The predominant contribution to the integral comes from the minimum I_-, since around I_+ the harmonic function $h^*_{B_-^N,B_+^N}(x)$ vanishes.

The proof will go in two steps. We define the tube in the z_0-direction,

$$\widetilde{C}_\delta \equiv \left\{z : \forall_{k\geq 1}|z_k| \leq \delta r_{k,N}/\sqrt{\lambda_{k,N}}\right\}, \tag{4.67}$$

and show that the mass of the complement of this tube is negligible. In a second step we show that within that tube, only the neighborhood of I_- gives a relevant and indeed the desired contribution. The reason for splitting our estimates up in this way is that we have to use different ways to control the non-quadratic terms.

Lemma 4.10. *Let $r_{k,N}$ be chosen as before and let $\delta = \sqrt{K\varepsilon|\ln\varepsilon|}$. Then there exists a finite numerical constant, A_7, such that for all N,*

$$\frac{1}{N^{N/2}}\int_{\widetilde{C}_\delta^c} e^{-G_{\gamma,N}(x)/\varepsilon}dx \leq A_7\sqrt{2\pi\varepsilon}^N\frac{e^{\frac{1}{4\varepsilon}}}{\sqrt{\det(\nabla F_{\gamma,N}(O))}}\varepsilon^K. \tag{4.68}$$

The same estimate holds for the integral over the complement of the set

$$D_\delta \equiv \{x : |z_0 - 1| \le \delta \vee |z_0 + 1| \le \delta\}. \tag{4.69}$$

Proof. Recall that $z_0 = \frac{1}{N}\sum_{i=0}^{N-1} x_i$. Then we can write

$$G_{\gamma,N}(x) = -\frac{1}{2}z_0^2 + \frac{1}{4}z_0^4 + \frac{1}{2N}(x, \mathbb{D}x) - \frac{1}{2N}\|x\|_2^2 + \frac{1}{2}z_0^2 + \frac{1}{4N}\|x\|_4^4 - \frac{1}{4}z_0^4. \tag{4.70}$$

Notice first that by applying the Cauchy-Schwartz inequality, it follows that

$$z_0^4 = N^{-4}\left(\sum_{i=0}^{N-1} x_i\right)^4 \le N^{-2}\left(\sum_{i=0}^{N-1} x_i^2\right)^2 \le N^{-1}\sum_{i=0}^{N-1} x_i^4. \tag{4.71}$$

Moreover, $N^{-1}\|x\|_2^2 = \|z\|_2^2$, so that expressed in the variables z,

$$\begin{aligned} G_{\gamma,N}(x) &\ge -\frac{1}{2}z_0^2 + \frac{1}{4}z_0^4 + \frac{1}{2N}(x, \mathbb{D}x) - \frac{1}{2}\sum_{k=1}^{N-1} z_k^2 \\ &= -\frac{1}{2}z_0^2 + \frac{1}{4}z_0^4 + \frac{1}{2}\sum_{k=1}^{N-1} \lambda_{k,N} z_k^2. \end{aligned} \tag{4.72}$$

Therefore, as in the estimate (4.40),

$$\begin{aligned} \int_{\widetilde{C}_\delta^c} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz &\le \int_{\mathbb{R}} e^{-\varepsilon^{-1}(z_0^4/4 - z_0^2/2)} dz_0 \sum_{k=1}^{N-1} \int_{|z_k| \ge \delta r_{k,N}/\sqrt{\lambda_{k,N}}} e^{-\lambda_{k,N}|z_k|^2/2\varepsilon} dz_k \\ &\quad \times \prod_{1 \le i \ne k \le N-1} \int_{\mathbb{R}} e^{-\lambda_{i,N}|z_i|^2/2\varepsilon} dz_i \qquad (4.73) \\ &\le \int_{\mathbb{R}} e^{-\varepsilon^{-1}(z_0^4/4 - z_0^2/2)} dz_0 \delta^{-1}\sqrt{\varepsilon} \sqrt{\prod_{i=1}^{N-1} 2\pi\varepsilon\lambda_{i,N}^{-1}} \sum_{k=1}^{N-1} r_{k,N}^{-1} e^{-\delta^2 r_{k,N}^2/2\varepsilon} \\ &\le \int_{\mathbb{R}} e^{-\varepsilon^{-1}(z_0^4/4 - z_0^2/2)} \sqrt{\prod_{i=1}^{N-1} 2\pi\varepsilon\lambda_{i,N}^{-1}} C\varepsilon^K. \end{aligned}$$

Since clearly,

$$\int_{\mathbb{R}} e^{-\varepsilon^{-1}(z_0^4/4 - z_0^2/2)} dz_0 = 2\sqrt{\pi\varepsilon} e^{1/4\varepsilon}(1 + \mathcal{O}(\varepsilon)), \tag{4.74}$$

this proves the first assertion of the lemma.

Quite clearly, the same bounds will show that the contribution from the set where $|z_0 \pm 1| \ge \delta$ are negligible, by just considering now the fact that the range of the integral over z_0 is bounded away from the minima in the exponent. □

Finally, we want to compute the remaining part of the integral in (4.66), i.e. the integral over $\widetilde{C}_\delta \cap \{x : |z_0 + 1| \le \delta\}$. Since the eigenvalues of the Hessian at I_-, $\nu_{k,N}$, are comparable to the eigenvalues $\lambda_{k,N}$ for $k \ge 1$ in the sense that there is a finite positive constant, c_μ^2, depending only on μ, such that $\lambda_{k,N} \le \nu_{k,N} \le c_\mu^2 \lambda_{k_N}$, and since $\nu_{0,N} = 2$, this set is contained in $C_{c_\mu\delta}$, where

$$C_\delta(I_-) \equiv \left\{ z \in \widehat{\mathbb{R}}^N : |z_0 + 1| \le \frac{\delta}{\sqrt{\nu_0}},\ |z_k| \le \delta\frac{r_{k,N}}{\sqrt{\nu_{k,N}}}\ 1 \le k \le N-1 \right\}. \tag{4.75}$$

It is easy to verify that on $C_\delta(I_-)$, there exists a constant, A_8, s.t.

$$\|z - z(I_-)\|_2^2 \leq \delta^2 \sum_{k=0}^{N-1} \frac{r_{k,N}^2}{\nu_{k,N}} \leq \delta^2 A_8 K_2^2. \tag{4.76}$$

and so, for $\delta = \sqrt{K\varepsilon |\ln \varepsilon|}$, $C_\delta(I_-) \subset z(B_-)$.

On $C_\delta(I_-)$ we have the following quadratic approximation.

Lemma 4.11. *For all N,*

$$\widetilde{G}_{\gamma,N}(z) + \frac{1}{4} - \frac{1}{2}\sum_{k=0}^{N-1} \nu_{k,N}|z_k|^2 = R(z) \tag{4.77}$$

and there exists a constant A_9 and δ_0 such that, for $\delta < \delta_0$, on $C_\delta(I_-)$

$$|R(z)| \leq A_9 \delta^3 \tag{4.78}$$

where the constants (ρ_k) are chosen as before such that $K_{4/3}$ is finite.

Proof. The proof goes in exactly the same way as in the previous cases and is left to the reader. □

With this estimate it is now obvious that

$$\begin{aligned}\int_{C_\delta(I_-)} \tilde{h}^*_{B_-^N, B_+^N}(z) e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz &= \int_{C_\delta(I_-)} e^{-\widetilde{G}_{\gamma,N}(z)/\varepsilon} dz \\ &= e^{1/4\varepsilon} \frac{\sqrt{2\pi\varepsilon}^N}{\sqrt{\det \nabla^2 F_{\gamma,N}(I_-)}} \left(1 + O(\delta^3/\varepsilon)\right),\end{aligned} \tag{4.79}$$

Using that $\tilde{h}^*_{B_-^N, B_+^N}(z)$ vanishes on B_+^N and hence on $C_\delta(I_+)$, this estimate together with Lemma 4.10 proves the proposition. □

4.3. **Proof of Theorem 3.1.**

Proof. The proof of Theorem 3.1 is now an obvious consequence of (2.16) together with Propositions 4.3 and 4.9. □

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